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The Forbes 400, the Pareto power-law and efficient markets

O.S. Klass[1], O. Biham[1,a], M. Levy[2], O. Malcai[1], and S. Solomon[1]

[1] Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel
[2] School of Business Administration, The Hebrew University, Jerusalem 91905, Israel

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Abstract. Statistical regularities at the top end of the wealth distribution in the United States are examined using the Forbes 400 lists of richest Americans, published between 1988 and 2003. It is found that the wealths are distributed according to a power-law (Pareto) distribution. This result is explained using a simple stochastic model of multiple investors that incorporates the efficient market hypothesis as well as the multiplicative nature of financial market fluctuations.

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1 Introduction

Once a year, the Forbes journal publishes the list of 400 richest people in the United States [1]. The list includes the net worth of each individual as well as background information about the businesses that have lead to this prosperity. It includes individuals involved in all sectors of the economy, such as computer software (Bill Gates, Paul Allen and Larry Ellison), financial investments (Warren Buffet), retailing (the Walton family), computer hardware (Michael Dell) as well as media, entertainment, communication, real estate and many other sectors.

Although the people in the Forbes 400 list made their fortunes in various different ways, the distribution of their wealths exhibits a striking statistical regularity. The wealths w_r of the 400 richest Americans in 2003, ordered by their ranks r, are shown in Figure 1, on a log-log scale. The data on this graph, known as the Zipf plot [2], can be fitted by a straight line, indicating that the wealths exhibit a power-law behavior of the form

$$w_r \sim r^{-\beta}, \quad (1)$$

where, for this data, the Zipf exponent is $\beta = 0.78 \pm 0.05$. This is a remarkable result because power-law distributions exhibit the special property that they have no characteristic scale. It may indicate that the same dynamical rules of gains/losses apply across the entire economy independently of the particular sector or the wealth and sophistication of different investors [3]. For example, if we hypothetically extrapolated the straight line fit, the wealth of the individual ranked number million, would be about three million US dollars. If this hypothesis is correct the Forbes data provides useful information about the wealths of people in percentiles far away from the top 400. These findings raise the broader, puzzling question about the origins of the anomalous nature of the wealth distribution. While the physical properties of humans (such as height) as well as the mental and social abilities typically follow Gaussian distributions, that tend to be rather narrow, their wealths are so widely distributed and span over seven orders of magnitude.

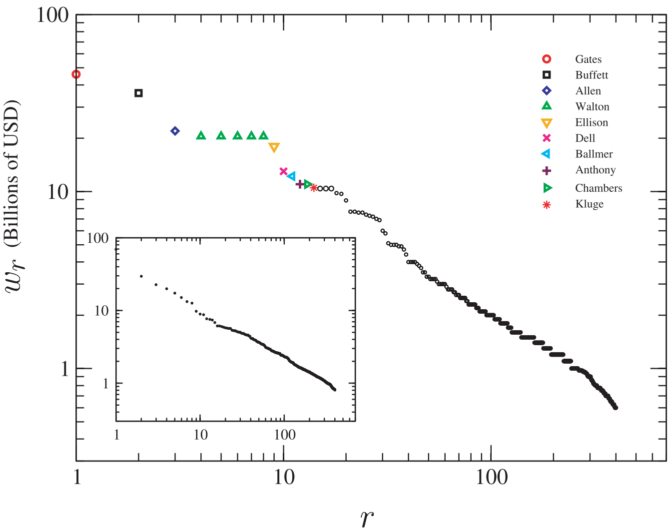

Fig. 1. Zipf plot of the wealths w_r of the investors in the Forbes 400 of 2003 vs. their ranks r. The power-law fit, with $\beta = 0.78 \pm 0.05$, was obtained in the range $10 \leq r \leq 300$. The corresponding simulation results are shown in the inset.

[a] e-mail: biham@phys.huji.ac.il

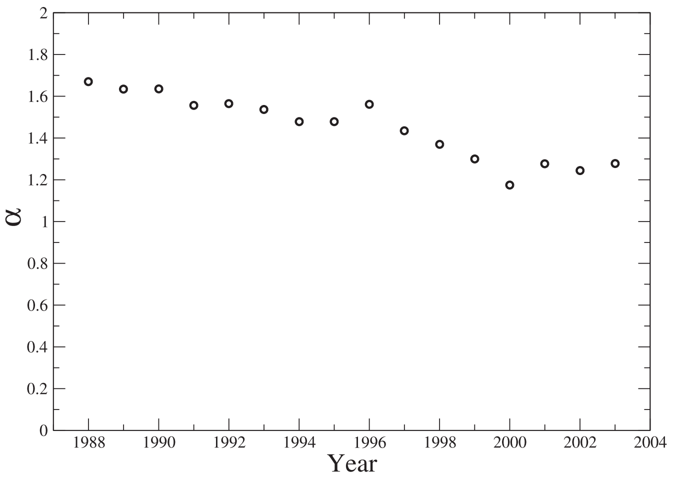

Fig. 2. The exponent $\alpha(t)$, obtained from the Zipf analysis, using the relation $\alpha = 1/\beta$, vs. the year, t, between 1988 and 2003. The power-law fit of the Zipf data was obtained in the range $10 \leq r \leq 300$.

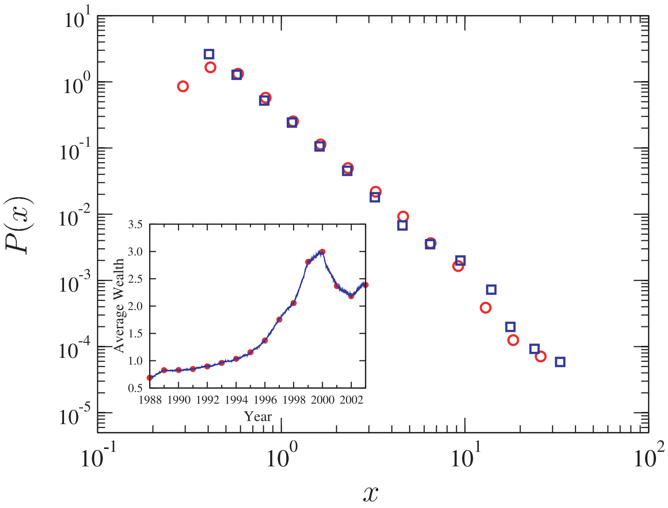

Fig. 3. The Pareto distribution $P(x)$ of the normalized wealths of the Forbes 400 investors (red circles) and the corresponding simulations results (blue squares). The inset shows the average wealth $\bar{w}(t)$ vs. time (circles), and simulation results with finer temporal resolution (rough solid line).

2 The Pareto distribution

The wealth-rank relation of equation (1) implies a power-law distribution of the wealth

$$P(w) \sim w^{-(\alpha+1)}, \tag{2}$$

often called the Pareto distribution [4], where α is the Pareto exponent. However, due to the relatively small number of data points for each year, the distribution $P(w)$ obtained after binning the data turns out to be rather noisy. Thus, for a single year, the Zipf plot and equation (1) provide more reliable results. Note that in general the connection between the Pareto and the Zipf exponents is $\alpha = 1/\beta$. Therefore, the distribution in Figure 1 corresponds to $\alpha = 1.28$. To examine the temporal variations of α we repeated the Zipf analysis for each year, from 1988 to 2003. It was found that α varies widely in the range between 1.1 and 1.7. The results for α vs. the year are shown in Figure 2.

Consider the average wealth

$$\bar{w}(t) = \sum_{r=1}^{400} w_r(t)/400 \tag{3}$$

of the 400 investors at year t. The time dependence of $\bar{w}(t)$ for the Forbes 400 investors between 1988 and 2003, shown in the inset of Figure 3 (circles), reflects the stock prices after the 1987 crash, through the recession of the early 1990's, the bubble economy of the late 1990's and its aftermath, and a recovery in 2003. Interestingly, there is a negative correlation between $\bar{w}(t)$ and $\alpha(t)$. In particular, $\alpha(t)$ is lowest (namely, inequality is highest) at the peak of the bubble in 2000, when $\bar{w}(t)$ is largest.

In order to obtain more reliable results from the Pareto analysis, we perform a multi-year analysis by combining the data from a sixteen-year period (1988–2003) to construct a much larger data set $w_r(t)$ where t is the year. The multi-year analysis requires using the normalized wealth variables $x_r(t) = w_r(t)/\bar{w}(t)$. The normalized variables from all years are combined together to form the probability density $P(x)$. It is found that in spite of the dramatic variations in the economic climate during this sixteen-year period, $P(x)$, shown in Figure 3 (circles), can be fitted very well by a power-law

$$P(x) \simeq k\ x^{-(\alpha+1)} \qquad x > x_{\min}, \tag{4}$$

characterized by a single exponent $\alpha = 1.49 \pm 0.04$, where $k = \alpha\ x_{\min}^{\alpha}$ is a normalization constant. The exponent α can be considered as an index of inequality in the society. For larger values of α the wealth is more evenly distributed, while for smaller values of α the gaps between rich and poor people broaden.

The power-law distribution of the wealths, obtained by the multi-year analysis of the Forbes data, confirms the hypothesis made over a century ago by the economist Vilfredo Pareto [4]. During the century since Pareto's work, empirical evidence has been accumulated in support of his hypothesis [5–9]. However, the underlying dynamics that leads to this broad and self-similar distribution has not yet been fully understood.

The fundamental property of financial markets that enables some people to get so rich is the multiplicative nature of capital investments. Due to this property, the effect of variations in stock prices on the wealth of each investor is proportional to the number of shares he/she holds. To exemplify this, consider an investor who has a capital of $1000. What does it take to make a million dollars out of it? In an additive investment process that yields a fixed amount of $1000 per year, it would take 999 years. However, a multiplicative process that doubles the investment each year would require only 10 years. Indeed, recent analysis of stochastic models with multiplicative dynamics

[10–13] has provided much insight about the origin of the power-law distributions in economic systems [14–20].

3 Stochastic multiplicative dynamics

Here we show that the wealth distribution obtained from the Forbes data can be reproduced by a simple stochastic-multiplicative model that incorporates the following assumptions: (I) the temporal variations in the market value of an investment portfolio can be described by random noise; (II) the markets are efficient [21,22], namely no investor can consistently "beat the market" reaping abnormal returns. In our model this feature is incorporated by drawing the random fluctuations of all the investor portfolio returns from the same probability distribution; (III) the random noise is of multiplicative nature, in the sense that the effect of stock price fluctuations on each individual is proportional to the number of stocks he/she owns; (IV) the distributions that appear in economic systems are bounded from below. For example, the income distribution is bounded from below due to social security policies that ensure some minimal income, to support the most basic needs for food and shelter. The lower bound is not fixed but proportional to some fraction of the average income, that reflects the cost of living. Similarly, the power-law distribution of the wealth exhibits a lower bound, $w_{\min}$, that may represent some minimal wealth needed for basic existence, while the excess beyond it tends to be invested. It is thus reasonable to assume that $w_{\min}$ is equal to some fraction $c < 1$ of the average wealth in the society.

Our model consists of N investors whose wealths at time t are given by $w_n(t)$, $n = 1, \ldots, N$, where the index n represents the investor's name. The wealths are updated asynchronously such that the average time between successive updates of each of the w_n's is Δt [14]. At each update, the randomly chosen wealth w_n is multiplied by a factor λ drawn from a given distribution $p(\lambda)$ [23]. As a result

$$w_n \to \lambda\, w_n, \tag{5}$$

while all the other w_m's remain unchanged. The threshold wealth $w_{\min}(t)$ required for entering and staying in the market at time t is given by $w_{\min}(t) = c\,\bar{w}(t)$, where $c < 1$ is a parameter and $\bar{w}(t)$ is the average wealth at time t. If $w_n(t)$ is reduced below $w_{\min}(t)$ the investor n is dropped from the list and a new investor then enters, taking over the index n, with an initial wealth $w_n(t) = w_{\min}(t)$.

We have performed computer simulations of the model for $N = 400$. The value of the parameter, $c = 0.337$, was obtained directly from the Forbes data, as the ratio $w_{\min}/\bar{w}$, averaged over the 16 years, where $w_{\min}$ is the wealth of the least wealthy investor on the list. Starting from a homogeneous distribution of the wealths, they spontaneously evolved towards a power-law distribution [shown in Fig. 3 (squares)], with an exponent α which is in excellent agreement with the empirical data. It was found that α varies from year to year but remains within the range of $1 < \alpha < 2$, in agreement with empirical studies of the wealth distributions in various countries [5–8].

The model predicts that the power-law distribution of the wealths extends well beyond the top 400 investors. To examine this feature we performed simulations with $N = 10\,000$ investors and obtained a power-law distribution of the wealths with the same value of α. Analysis of the model shows that for N not too small, the exponent α is determined by the parameter c, according to [14,18]

$$\alpha \simeq \frac{1}{1-c}. \tag{6}$$

This result provides a strong connection between the lower-cutoff, that is typically determined by certain social security policies, and the exponent α that affects the wealth distribution of the entire population, including those at the top. The dependence of α on c given by equation (6) indeed confirms that $\alpha > 1$ for $c < 1$, in agreement with empirical studies [5–8]. In power-law distributions with $1 < \alpha < 2$, the first moment remains bounded when the system size increases, while the second moment diverges. In case that $\alpha > 2$, the second moment is bounded as well. Such power-law distributions, with a bounded second moment, are obtained when the model is simulated with $c > 0.5$ [18], in agreement with equation (6). In empirical studies one obtains $\alpha < 2$, which can be related to the fact that c is significantly smaller than 0.5.

It turns out that power-laws that emerge in economic systems as well as in general multiplicative processes are non-universal, unlike the critical exponents that appear in second order phase transitions, e.g., in magnets and fluids [24]. One may wonder why the exponent α of the wealth distribution tends to take values in the range between 1 and 2, while in other physical systems there is no such restriction. For example, the distribution of magnitudes of earthquakes follows a power-law with $\alpha < 1$ [25]. In power-law distributions with $\alpha < 1$ the first moment diverges as the system increases. In the case of earthquakes this does not lead to a contradiction, due to the unlimited separation of time scales between the slow process of strain buildup and its extremely fast release in an earthquake. Since the time between two large earthquakes diverges with their size, they have unlimited time to accumulate the energy. On the contrary, in economic systems the processes of wealth accumulation and redistribution take over comparable time scales. Therefore $\alpha < 1$ cannot be obtained. It is reasonable to believe that the average wealth in a society is tied in some way to the average productivity per worker. This productivity is presumably bounded and cannot diverge as the system size increases. Note that in the model, $\alpha < 1$ can be obtained for small systems when c is reduced below some threshold. This leads to large fluctuations, that may characterize economic instability [18].

4 Efficient markets

The power-law distribution of the wealths is insensitive to the properties of the distribution $p(\lambda)$. However, a crucial assumption in the model is that the same distribution $p(\lambda)$

is used for all the investors. It was found that simulations in which this assumption is violated do not give rise to a homogeneous power-law distribution of the wealths [8,26]. Thus, according to the model, the power-law wealth distribution in the Forbes data indicates that the short time gain/loss distribution is similar for all the investors. This conclusion may look like a paradox. While it is clear that to enter the Forbes 400 list one has to do 'something right' in a big and consistent way, the statistical regularities of the wealth distribution can be captured by a model that exhibits a completely random dynamics. The resolution of this paradox is that although the distribution $p(\lambda)$ is similar for all the investors, the actual sums of λ's that each one draws are different. The multiplicative dynamics greatly magnifies the differences between more successful and less successful investors, and builds up the power-law distribution of their wealths. The assumption that $p(\lambda)$ is the same for all the investors is a strict implementation of the efficient market hypothesis, which states that no investor can consistently obtain a return distribution better (adjusted for risk) than the return distributions of the other investors. Thus, our model provides a connection between the efficient market hypothesis and the Pareto distribution of wealth.

5 Temporal variations, gains and losses

The multi-year data set allows us to examine the temporal variations of the wealths, and particularly the distribution of returns of individual investors. The log-return of investor n in the period between year t and year $t + \tau$, is given by

$$g_n(\tau) = \ln w_n(t + \tau) - \ln w_n(t). \quad (7)$$

From the Forbes 400 data we calculated $g_n(\tau)$ for all the pairs of years t and $t + \tau$ where $\tau = 1$, 2, 4 and 8 years, for those investors which were included in the list in both years. Using these results we obtained the distributions $P_\tau(g)$ of the log-returns, shown in Figure 4.

In order to compare these results to the temporal fluctuations in the model one has to relate the time units of the model to the physical time. It turns out that the model exhibits remarkable scaling properties that are useful for this task. Due to the scaling, the results of simulations are invariant to the choice of the time unit Δt of the model, as long as the width of $p(\lambda)$ is adjusted accordingly. In the simulations, we chose $\Delta t =$ one day, assuming 250 working days a year. Note that $g_n(\tau)$ for $\tau = 1$ day, coincides with the single-day log-return given by $\ln \lambda$. As a result, for $\tau = 1$ day $P_\tau(g) = P(\ln \lambda) = \lambda p(\lambda)$. Therefore, $P_\tau(g)$ depends on the detailed form of $p(\lambda)$ as long as τ is small. However, the Forbes data analyzed here is published only once a year (corresponding to $\tau = 250$). Over such a period the fine details of $p(\lambda)$ are washed out except for its average $\langle\lambda\rangle$ and standard deviation.

In the simulations, the standard deviation of $p(\lambda)$, that controls the mobility of people across the wealth distribution, was adjusted to match the empirical result that each year about 15 percent of the list members are replaced by others. The average $\langle\lambda\rangle(t)$ that determines the time evolution of $\bar{w}(t)$ was adjusted in time to follow the evolution of the empirical $\bar{w}(t)$ (see inset of Fig. 3). The distributions of the log-returns obtained from the simulations are shown in Figure 4 (inset).

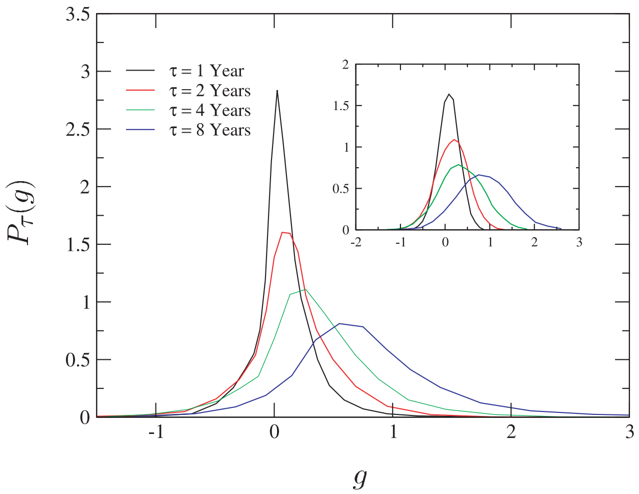

Fig. 4. The probability distribution of the returns $P_\tau(g)$, $\tau = 1, 2, 4$ and 8 years, for the Forbes investors. Corresponding simulation results are shown in the inset.

The temporal resolution of the Forbes data is one year. However, the scaling properties of the model allow us to study fluctuations over shorter time scales. In particular, it is interesting to explore the fluctuations of $\bar{w}$, which are expected to be strongly related to those of market indices such as the Standard & Poor 500 (S&P500). It was observed long ago that fluctuations in financial markets exhibit non-Gaussian "fat tailed" distributions. Already in the 1960's Mandelbrot studied the fluctuations in cotton prices and discovered that they can be expressed by a Lévy-stable distribution [27]. Recent empirical studies of the fluctuations of the S&P500 index provided apparently conflicting results. The central peak of the distribution of returns, $P(0)$, was found to exhibit scaling behavior which is consistent with the Lévy distribution [28]. However, the tails of the distribution of returns were found to decay as a power-law with $\alpha \simeq 3$, namely beyond the Lévy-stable range, $0 < \alpha < 2$ [29].

An interesting question is whether the model, studied here in the economic context of the wealth distribution, can reproduce the empirical results for the fluctuations in financial markets. To study this question we have analyzed the temporal fluctuations of $\bar{w}$ obtained from the model. The log-return $G(\tau)$ for $\bar{w}$ over a time interval τ is given by

$$G(\tau) = \ln \bar{w}(t + \tau) - \ln \bar{w}(t). \quad (8)$$

The distribution $P_\tau(G)$ of the log-returns shows a Lévy-like scaling of the central peak $P_\tau(0) \sim \tau^{-1/\alpha}$, characterized by the same exponent $\alpha = 1.49 \pm 0.04$, found for the wealth distribution (Fig. 5), which is close to the value found for the S&P500 index [28]. However, the tails are found to decay according to $P(g) \sim |g|^{-(\alpha+1)}$, where

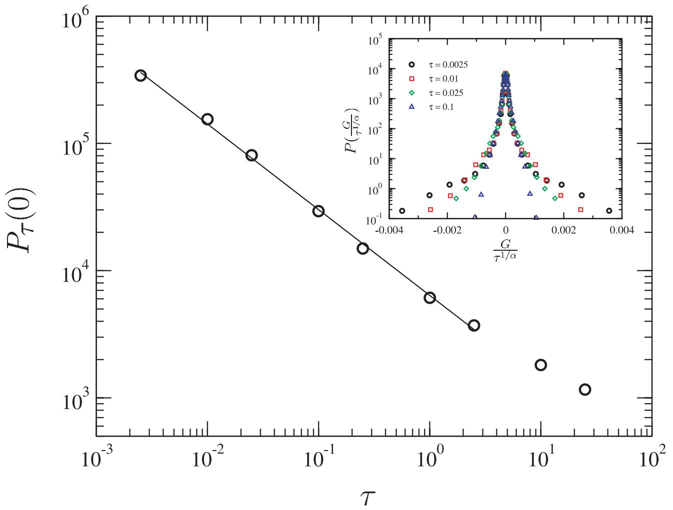

Fig. 5. The height of the central peak of $P_\tau(G)$ vs. τ (in units of Δt) on a log-log scale. The inset shows a scaling plot of $P_\tau(r/\tau^{1/\alpha})$ for $\tau = 1, 2, 4, 8$. The central peaks coincide confirming the scaling in the main Figure but the tails deviate from the Lévy-stable form.

$\alpha = 2.3$, namely falls outside the range of the Lévy-stable distribution [30]. These results are consistent with the empirical finding both for the central peak [28] and for the tails [29].

6 Summary and discussion

We have presented a simple stochastic model of multiple investors that incorporates the efficient market hypothesis as well as the multiplicative nature of financial market fluctuations. The model exhibits a power-law (Pareto) distribution of the wealths of the individual investors, in good agreement with empirical results, as revealed by the Forbes 400 lists. It thus provides an interesting connection between market efficiency, the Pareto distribution of wealth and the non-Gaussian fluctuations in financial markets.

The observed regularity of the wealth distribution at the high wealth range confirms the hypothesis made over a century ago by Pareto [4]. The fact that power-law distributions exhibit no characteristic scale indicates that the same dynamical rules of gains/losses may apply independently of the particular wealth percentile. The power-law distribution may thus extend to people in wealth percentiles far away from the top 400. Note, however, that empirical studies indicate that the power-law distribution of wealth does not extend to the entire population but exhibits a cross-over to a different distribution at lower percentiles [31]. This may be explained by the fact that people in lower percentiles make very few transactions in the stock market and their wealth is only marginally affected by financial market fluctuations. It will thus be useful to develop dynamical models that account for both the power-law distribution of wealth at the higher percentiles and the exponential distribution at the lower percentiles.

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References

1. For the lists from recent years see www.forbes.com/lists
2. G.K. Zipf, *Humen Behavior and the Principle of least Effort* (Addison-Wesley Press, Cambridge, MA, 1949)
3. P.W. Anderson, *Some Thoughts about Distributions in Economics*, in *The Economy as an Evolving Complex System II*, edited by W.B. Arthur, S.N. Durlauf, D.A. Lane, SFI Studies in the Sciences of Complexity (Addison Wesley Longman, 1997)
4. V. Pareto *Cours d'Économique Politique* (Macmillan, Paris), Vol. 2 (1897)
5. J. Steindl, *Random Processes and the Growth of Firms - A Study of the Pareto Law* (Charles Griffin and Company, London, 1965)
6. A.B. Atkinson, A.J. Harrison, *Distribution of Total Wealth in Britain* (Cambridge University Press, Cambridge, 1978)
7. J. Persky, J. Economic Perspectives **6**, 181 (1992)
8. M. Levy, in *The Economy as an Evolving Complex System III,* edited by S. Durlauf, L. Blume (Oxford University Press, 2003)
9. O.S. Klass, O. Biham, M. Levy, O. Malcai, S. Solomon, Economics Lett. **90**, 290 (2006)
10. D.G. Champernowne, Econometrica **63**, 318 (1953)
11. H. Wold, P. Whittle, Econometrica **25**, 591 (1957)
12. H.A. Simon, C.P. Bonini, Am. Econ. Rev. **48**, 607 (1958)
13. H. Kesten, Acta. Math. **131**, 207 (1973)
14. M. Levy, S. Solomon, Int. J. Mod. Phys. C **7**, 595 (1996)
15. D. Sornette, R. Cont, J. Phys. I France **7**, 431 (1997)
16. H. Takayasu, A.-H. Sato, M. Takayasu, Phys. Rev. Lett. **79**, 966 (1997)
17. M. Marsili, S. Maslov, Y.-C. Zhang, Physica A **253**, 403 (1998)
18. O. Malcai, O. Biham, S. Solomon, Phys. Rev. E **60**, 1299 (1999)
19. J.-P. Bouchaud, M. Mézard, Physica A **282**, 536 (2000)
20. J. Silver, E. Slud, K. Takamoto, J. Econ. Theory **106**, 417 (2002)
21. E.F. Fama, J. Finance **25**, 383 (1970)
22. E.F. Fama, J. Finance **46**, 1575 (1991)
23. The results presented here were obtained using a simple bimodal distribution in which λ can take one of two values: $\lambda_1 = 1 + \Delta\lambda_1$ and $\lambda_2 = 1 - \Delta\lambda_2$, where $p(\lambda_1) = p(\lambda_2) = 1/2$ and $\Delta\lambda_1, \Delta\lambda_2 \ll 1$. Using other forms of $p(\lambda)$ such as the Gaussian distribution, which exhibit finite values of the variance, we found that the Pareto exponent does not depend on $p(\lambda)$
24. H.E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford University Press, Oxford, 1971)
25. B. Gutenberg, C.F. Richter Ann. Geophys. **9**, 1 (1956)
26. M. Levy J. Economic Theory **110**, 42 (2003)
27. B.B. Mandelbrot, J. Business **36**, 394 (1963)
28. R.N. Mantegna, H.E. Stanley, Nature **376**, 46 (1995)
29. P. Gopikrishnan, V. Plerou, L.A.N. Amaral, M. Meyer, H.E. Stanley, Phys. Rev. E **60**, 5305 (1999)
30. O. Biham, Z.-F. Huang, O. Malcai, S. Solomon Phys. Rev. E **64**, 026101 (2001)
31. A. Drăgulescu, V.M. Yakovenko, Physica A **299**, 213 (2001)